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# The $C^*$-algebra of a vector bundle and fields of Cuntz algebras

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**Abstract**

We study the Cuntz–Pimsner algebra associated with the module of continuous sections of a Hilbert bundle, and prove that it is a continuous bundle of Cuntz algebras. Furthermore, we assign to bundles of Cuntz algebras carrying a global circle action a class in the representable *KK*-group of the zero-grade bundle. We explicitly compute such class for the Cuntz–Pimsner algebra of a vector bundle.


*Keywords:* Cuntz–Pimsner algebra; Cuntz algebra; Continuous field; Vector bundle; *K*-theory

## 1. Introduction

The Cuntz–Pimsner algebras (CP-algebras in the sequel, also called Cuntz–Krieger–Pimsner algebras) were introduced in [23], as a natural generalization of the Cuntz–Krieger algebras [8,10] and crossed products by a single automorphism. They also include crossed products by partial automorphisms and by Hilbert bimodules in the sense of [17,1]. A categorical approach to the CP-algebra has been performed in [13].

Let $X$ be a locally compact Hausdorff space. In the present paper, we proceed to the study of the CP-algebra associated with the module of continuous sections of a Hilbert bundle $\mathcal{H} \to X$, started in [26] in a categorical framework (in the sequel, we

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will more concisely say *the CP-algebra of the Hilbert bundle*). The interest in such algebras arises for their role in the context of crossed products by endomorphisms, and duality for non-compact groups (see [27]). Anyway, they are an interesting class also from the viewpoint of the classification of $C^*$-algebras by *KK*-theoretical invariants, in the spirit of [21,7]. We will develop here the basic properties of such CP-algebras, and give some applications for Cuntz algebra bundles; more detailed *K*-theoretical and classification questions will be approached in future papers. The present work is organized as follows:

In Section 2, we investigate the basic properties of the CP-algebra $\mathcal{O}_{\mathcal{H}}$ of a Hilbert bundle $\mathcal{H} \to X$. We prove that $\mathcal{O}_{\mathcal{H}}$ is a continuous bundle, having as fibre the Cuntz algebra $\mathcal{O}_d$ (Proposition 2).

In Section 3, we consider graded $\mathcal{O}_d$-bundles (i.e., locally trivial continuous bundles of Cuntz algebras with a global circle action), and assign to them an invariant belonging to the representable Kasparov group of the zero grade $C^*$-algebra (Eq. (7)). Such invariant is used in a Pimsner–Voiculescu exact sequence for the *KK*-theory of the $\mathcal{O}_d$-bundle (8), and is computed for the CP-algebra of a vector bundle (Theorem 5). As an application, we prove that a stable graded isomorphism at level of the CP-algebra implies an equivalence in terms of (representable) *K*-theory of the underlying vector bundles (Proposition 7). Viceversa, if the base space is a finite *CW*-complex, the equivalence in *K*-theory of the vector bundles implies a (stable) isomorphism of the CP-algebras (Proposition 10).

### *1.1. Preliminaries*

The notion of continuous field (bundle) of $C^*$-algebras (resp. Banach spaces, Hilbert spaces) is well known in literature; for basic notions and terminology we refer to [11, Section 10; 22]. If $\mathcal{A}$ is a $C^*$-algebra, we call *$\mathcal{A}$-bundle* a locally trivial continuous bundle of $C^*$-algebras with fibre $\mathcal{A}$. In the present paper, we will also consider continuous fields of Hilbert (Banach) spaces having finite-dimensional fibres; we call *Hilbert* (*Banach*) *bundles* the corresponding *espaces fibrés vectoriels* in the sense of [12, I.2], according to the terminology of Dupré [15,16]. In particular, a vector bundle (i.e., a locally trivial Hilbert bundle) will be denoted by the usual notation $p : \mathcal{E} \to X$. For basic properties and terminology about vector bundles we refer to [2,19].

For basic notions and properties of $C_0(X)$-algebras, we refer to [20,6]. If $\mathcal{A}$ is a $C_0(X)$-algebra, we denote by $\mathbf{aut}_X\mathcal{A}$ (resp. $\mathbf{end}_X\mathcal{A}$) the set of $C_0(X)$-automorphisms (resp. $C_0(X)$-endomorphisms) of $\mathcal{A}$; moreover, we denote by $\otimes_X$ the minimal $C_0(X)$-algebra tensor product [5].

Let $X$ be a locally compact Hausdorff space, $\mathcal{A}$, $\mathcal{B}$ $C_0(X)$-algebras. A *$C_0(X)$-Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule* is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$ such that $(af)\psi b = a\psi(fb)$ for every $f \in C_0(X)$, $\psi \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$. We denote by $L(\mathcal{M})$ the $C_0(X)$-algebra of bounded, right $\mathcal{B}$-module operators on $\mathcal{M}$, and by $K(\mathcal{M}) \subseteq L(\mathcal{M})$ the $C_0(X)$-algebra of compact operators on $\mathcal{M}$.

Let $\mathcal{M}$ be a $C_0(X)$-Hilbert $\mathcal{A}$-bimodule. If $\alpha : \mathcal{A} \to \mathcal{B}$ is a $C_0(X)$-algebra isomorphism, then a natural structure of $C_0(X)$-Hilbert $\mathcal{B}$-bimodule can be assigned to $\mathcal{M}$:

$$\psi, b \mapsto \psi\alpha^{-1}(b), \quad b, \psi \mapsto \alpha^{-1}(b)\psi, \quad \left\langle \psi, \psi' \right\rangle_\alpha := \alpha\left(\left\langle \psi, \psi' \right\rangle\right),$$

$\psi, \psi' \in \mathcal{M}$, $b \in \mathcal{B}$. We call the corresponding $\mathcal{B}$-bimodule *pullback bimodule*, and denote it by $\mathcal{M}_\alpha$. Suppose now $\mathcal{A} = \mathcal{B}$, $\alpha \in \mathbf{aut}_X\mathcal{A}$; if $\mathcal{M}$ is finitely generated and projective as a right Hilbert $\mathcal{A}$-module (i.e. $\mathcal{M} \subseteq \mathcal{A}^n$, so that every $\psi \in \mathcal{M}$ can be regarded as an $n$-ple $(\psi_h)$, $\psi_h \in \mathcal{A}$, $h = 1, \ldots, n$), then $\mathcal{M}_\alpha$ can be concretely realized as the set $\mathcal{M}_\alpha := \left\{(\alpha^{-1}(\psi_h)), (\psi_h) \in \mathcal{M}\right\}$, endowed with the above bimodule structure. There is an isomorphism of $C_0(X)$-Hilbert $\mathcal{A}$-bimodules

$$U : \mathcal{M} \to \mathcal{M}_\alpha, \quad U(\psi_h) := (\alpha^{-1}(\psi_h)). \tag{1}$$

For basic notions and terminology about *KK*-theory, we refer to [4, Section 17]. If $\mathcal{A}$, $\mathcal{B}$ are $C_0(X)$-algebras, we denote by $KK(X; \mathcal{A}, \mathcal{B})$ the *representable KK*-group introduced in [20] (note that the notation $\mathcal{R}KK(X; \mathcal{A}, \mathcal{B})$ is used in the above reference). Kasparov cycles giving rise to elements of $KK(X; \mathcal{A}, \mathcal{B})$ will be denoted as pairs $(\mathcal{M}, F)$, where $\mathcal{M}$ is a ($\mathbb{Z}_2$-graded) $C_0(X)$-Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule and $F \in L(\mathcal{M})$ is a ‘Fredholm operator’ (with degree one). In order for more concise notations, for every $C_0(X)$-algebra $\mathcal{A}$ we write $KK(X; \mathcal{A}) := KK(X; \mathcal{A}, \mathcal{A})$. The Kasparov product [20, Section 2.21] induces a natural ring structure on $KK(X; \mathcal{A})$, having as identity the class $[1]_\mathcal{A}$ associated with the obvious bimodule structure on $\mathcal{A}$. We also define

$$RK^0(X) := KK(X; C_0(X)).$$

If $X$ is compact, it is verified that $RK^0(X)$ coincides with the usual *K*-theory group $K^0(X)$ (see [20, 2.19]). Let $\mathcal{E} \to X$ be a rank $d$ vector bundle, $d \in \mathbb{N}$; we denote by $\widehat{\mathcal{E}}$ the Hilbert $C_0(X)$-bimodule of continuous, vanishing at infinity section of $\mathcal{E}$, endowed with the left $C_0(X)$-action coinciding with the right one.

**Lemma 1.** *Let $X$ be a $\sigma$-compact Hausdorff space, $\mathcal{E} \to X$ a vector bundle. Then, $C_0(X)$ acts on the left over $\widehat{\mathcal{E}}$ by elements of $K(\widehat{\mathcal{E}})$, and the pair $(\widehat{\mathcal{E}}, 0)$ is a Kasparov module with class $[\mathcal{E}] := [(\widehat{\mathcal{E}}, 0)] \in RK^0(X)$.*

**Proof.** We denote by $1 \in L(\widehat{\mathcal{E}})$ the identity on $\widehat{\mathcal{E}}$, and by $\theta_{\psi,\psi'} \in K(\widehat{\mathcal{E}})$, $\psi, \psi' \in \widehat{\mathcal{E}}$, the operator $\theta_{\psi,\psi'}(\varphi) := \psi\left\langle \psi', \varphi \right\rangle$. Since $X$ is $\sigma$-compact, there is a sequence $\{K_n\}_n$ of compact subsets covering $X$. We consider a partition of unity $\left\{\lambda_n\right\}$ such that $\overline{\mathrm{supp}\,\lambda_n} = K_n$, $n \in \mathbb{N}$. By the Serre–Swan theorem, the bimodule of continuous sections of the restriction $\mathcal{E}|_{K_n}$ is finitely generated by a set $\left\{\varphi_{n,k}\right\}_k$; we define $\psi_{n,k} := \lambda_n\varphi_{n,k} \in \widehat{\mathcal{E}}$. Furthermore, let $u_n := \sum_k \theta_{\psi_{n,k},\psi_{n,k}} \in K(\widehat{\mathcal{E}})$; then by construction $u_n = \lambda_n^2$ (in fact, $\sum_k \theta_{\varphi_{n,k},\varphi_{n,k}} = 1|_{K_n}$) and $\sum_n^m u_n = \sum_n^m \lambda_n^2 \overset{m}{\to} 1$ (in the strict topology). Thus, we conclude that $\widehat{\mathcal{E}}$ is countably generated by the set $\left\{\psi_{n,k}\right\}$. Furthermore, if $f \in C_0(X)$ then $\left\| f - f\sum_n^m u_n \right\| = \left\| f - \sum_n^m \lambda_n^2 f \right\| \stackrel{m}{\to} 0$; thus $f$ is norm limit of elements of $K(\widehat{\mathcal{E}})$, and $C_0(X)$ acts on the left over $\widehat{\mathcal{E}}$ by elements of $K(\widehat{\mathcal{E}})$. We conclude that the pair $(\widehat{\mathcal{E}}, 0)$ defines a class in $RK^0(X)$. $\square$

Let now $\mathcal{A}$ be a $\sigma$-unital $C_0(X)$-algebra; the algebraic tensor product $\widehat{\mathcal{E}} \odot_{C_0(X)} \mathcal{A}$ with coefficients in $C_0(X)$ is endowed with a natural $\mathcal{A}$-valued scalar product $\left\langle \psi \otimes a, \psi' \otimes a' \right\rangle := \left\langle \psi, \psi' \right\rangle \cdot a^* a'$, $\psi, \psi' \in \widehat{\mathcal{E}}$, $a, a' \in \mathcal{A}$. We denote by $\widehat{\mathcal{E}} \otimes_X \mathcal{A}$ the corresponding completion. $\widehat{\mathcal{E}} \otimes_X \mathcal{A}$ is a $C_0(X)$-Hilbert $\mathcal{A}$-bimodule in the natural way (in particular, the left $\mathcal{A}$-action is defined by $a', \psi \otimes a \mapsto \psi \otimes (a'a)$, $\psi \in \widehat{\mathcal{E}}$, $a, a' \in \mathcal{A}$). Since $\mathcal{A}$ is $\sigma$-unital, $\mathcal{M} := \widehat{\mathcal{E}} \otimes_X \mathcal{A}$ is countably generated. By the previous lemma, $\mathcal{A}$ acts on the left on $\mathcal{M}$ by elements of $K(\mathcal{M})$, so that the pair $(\mathcal{M}, 0)$ defines a class in $KK(X; \mathcal{A})$, corresponding to the Kasparov product $[\mathcal{E}] \cdot [1]_{\mathcal{A}}$. Thus, the 'extending the scalars' operation $\widehat{\mathcal{E}} \mapsto \widehat{\mathcal{E}} \otimes_X \mathcal{A}$ defines a morphism

$$i_{\mathcal{A}} : RK^0(X) \to KK(X; \mathcal{A}) \tag{2}$$

(see [20, 2.19]). The morphism $i_{\mathcal{A}}$ is natural, in the following sense: if $\alpha : \mathcal{A} \to \mathcal{B}$ is a $C_0(X)$-algebra isomorphism, then by functoriality a ring isomorphism $\alpha_* : KK(X; \mathcal{A}) \to KK(X; \mathcal{B})$ is defined:

$$i_{\mathcal{B}}[\mathcal{E}] := [\widehat{\mathcal{E}} \otimes_X \mathcal{B}, 0] = [\widehat{\mathcal{E}} \otimes_X \alpha(\mathcal{A}), 0] = [(\widehat{\mathcal{E}} \otimes_X \mathcal{A})_\alpha, 0] = \alpha_* i_{\mathcal{A}}[\mathcal{E}]. \tag{3}$$

We will also make use of a second type of tensor product of $\widehat{\mathcal{E}}$ by $\mathcal{A}$. As first, we note that $\widehat{\mathcal{E}}$ has an obvious structure of $C_0(X)$-Hilbert $K(\widehat{\mathcal{E}})$-$C_0(X)$-bimodule; we denote by

$$\widehat{\mathcal{E}} \widehat{\otimes}_X \mathcal{A} \tag{4}$$

the external tensor product of Hilbert bimodules with coefficients in $C_0(X)$. $\widehat{\mathcal{E}} \widehat{\otimes}_X \mathcal{A}$ is a $C_0(X)$-Hilbert $(K(\mathcal{E}) \otimes_X \mathcal{A})$-$\mathcal{A}$-bimodule in the natural way; note that $\widehat{\mathcal{E}} \widehat{\otimes}_X \mathcal{A}$ is isomorphic to $\widehat{\mathcal{E}} \otimes_X \mathcal{A}$ as a *right* Hilbert $\mathcal{A}$-module.

We recall that if $\mathcal{M}$ is a Hilbert $\mathcal{A}$-bimodule, then the CP-algebra $\mathcal{O}_{\mathcal{M}}$ is endowed with the circle action $\gamma : \mathbb{T} \to \mathbf{aut}\mathcal{O}_{\mathcal{M}}$, $\gamma_z(\psi) := z\psi$, $\psi \in \mathcal{M}$, $z \in \mathbb{T}$ (see [23, Section 3; 13, Section 3]). We also note that by universality, for every pullback bimodule $\mathcal{M}_\alpha$ there is an isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \mathcal{O}_{\mathcal{M}_\alpha}$ preserving the $\mathbb{Z}$-grading induced by the circle action.

Let $\mathcal{A}$ be a $C^*$-algebra carrying an action $\alpha : \mathbb{T} \to \mathbf{aut}\mathcal{A}$. Then, every spectral subspace $\mathcal{A}^k := \left\{ a \in \mathcal{A} : \alpha_z(a) = z^k a, z \in \mathbb{T} \right\}$, $k \in \mathbb{Z}$, is a Hilbert bimodule over the zero-grade algebra $\mathcal{A}^0$, endowed with the structure

$$a, b \mapsto ab, \quad b, a \mapsto ba, \quad \left\langle b, b' \right\rangle := b^* b' \in \mathcal{A}^0,$$

$a \in \mathcal{A}^0$, $b, b' \in \mathcal{A}^k$. $(\mathcal{A}, \mathbb{T}, \alpha)$ is said *semi-saturated* if $\mathcal{A}$ is generated as a $C^*$-algebra by $\mathcal{A}^0$, $\mathcal{A}^1$ [1,17]; if $\mathcal{A}^1$ is full (i.e. $(\mathcal{A}^1)^*\mathcal{A}^1 = \mathcal{A}^0$), it is easily verified that there is an isomorphism $K(\mathcal{A}^k) \simeq \mathcal{A}^0$, $k \in \mathbb{Z}$, i.e. every $\mathcal{A}^k$ is an imprimitivity $\mathcal{A}^0$-bimodule (in the sense of [3]). If $(\mathcal{B}, \beta, \mathbb{T})$ is a $C^*$-dynamical system, we say that a morphism $\phi : \mathcal{A} \to \mathcal{B}$ is *graded* if $\phi(\mathcal{A}^k) \subseteq \mathcal{B}^k$, $k \in \mathbb{Z}$, and use the notation $\phi : (\mathcal{A}, \mathbb{Z}) \to (\mathcal{B}, \mathbb{Z})$. If $(\mathcal{A}, \mathbb{T}, \alpha)$ is semi-saturated and $\mathcal{A}^1$ is full, then by universality of the CP-algebra there is an isomorphism $(\mathcal{A}, \mathbb{Z}) \to (\mathcal{O}_{\mathcal{A}^1}, \mathbb{Z})$, where $\mathcal{O}_{\mathcal{A}^1}$ is the CP-algebra of $\mathcal{A}^1$ regarded as a Hilbert $\mathcal{A}^0$-bimodule (see [1, Theorem 3.1; 23, Theorem 3.12]).

## 2. CP-algebras and bundles

Let $\mathcal{O}_d$, $d \in \mathbb{N}$, denote the Cuntz algebra generated by a multiplet of orthogonal isometries $\{\psi_i\}_{i=1}^d$. It is well known that the circle action $\gamma : \mathbb{T} \to \mathbf{aut}\mathcal{O}_d$, $\gamma_z(\psi_i) := z\psi_i$, is semi-saturated; thus, $\mathcal{O}_d$ is endowed with a $\mathbb{Z}$-grading, and each spectral subspace $\mathcal{O}_d^k$, $k \in \mathbb{Z}$, can be regarded as an imprimitivity bimodule over the UHF-algebra $\mathcal{O}_d^0 \simeq \mathrm{UHF}(d^\infty)$. Moreover, every $\mathcal{O}_d^k$ is an inductive limit $\lim_{\to r} \mathbb{M}_{d^r, d^{r+k}}$, where the Banach space $\mathbb{M}_{d^r, d^{r+k}}$ of $d^r \times d^{r+k}$-matrices is embedded in $\mathbb{M}_{d^{r+1}, d^{r+1+k}}$ by tensoring on the right by the identity $1_d \in \mathbb{M}_d$ (see [14]).

Let $X$ be a locally compact Hausdorff space, $\mathcal{H} \to X$ a Hilbert bundle with rank $d : X \to \mathbb{N}$. For every $x \in X$, we denote by $\mathcal{H}_x \simeq \mathbb{C}^{d(x)}$ the fibre of $\mathcal{H}$ over $x$. We denote by $\widehat{\mathcal{H}}$ the Hilbert $C_0(X)$-bimodule of continuous, vanishing at infinity sections of $\mathcal{H}$, with left $C_0(X)$-action coinciding with the right one. We make the standard assumptions that $\mathcal{H}$ is full as a Hilbert bundle (see [15]), and that $\widehat{\mathcal{H}}$ is full as a right Hilbert $C_0(X)$-module.

We denote by $\mathcal{O}_\mathcal{H}$ the CP-algebra of $\widehat{\mathcal{H}}$. In particular, the analogous notations will be used for a vector bundle $\mathcal{E} \to X$. Since $\mathcal{O}_\mathcal{H}$ is generated by $\widehat{\mathcal{H}} \subset \mathcal{O}_\mathcal{H}^1$, it turns out that the canonical circle action defined on $\mathcal{O}_\mathcal{H}$ is semi-saturated.

For every $r, s \in \mathbb{N}$, we denote by $(\mathcal{H}^r, \mathcal{H}^s)$ the Banach $C_0(X)$-bimodule of compact $C_0(X)$-module operators from the (internal) tensor power $\widehat{\mathcal{H}}^r$ into $\widehat{\mathcal{H}}^s$ (so that $(\mathcal{H}^r, \mathcal{H}^s) \simeq \widehat{\mathcal{H}}^s \otimes_X \widehat{\mathcal{H}}^{*r}$, where $\widehat{\mathcal{H}}^*$ is the conjugate bimodule). We also adopt the notation $\iota := \mathcal{H}^0 := X \times \mathbb{C}$, so that there are natural identifications $(\iota, \iota) = C_0(X)$, $\widehat{\mathcal{H}} \simeq (\iota, \mathcal{H})$, $\widehat{\mathcal{H}}^r \simeq (\iota, \mathcal{H}^r)$. Every $(\mathcal{H}^r, \mathcal{H}^s)$ corresponds to the module of continuous, vanishing at infinity sections of a Banach bundle, namely $\mathcal{H}^s \otimes \mathcal{H}^{*r} \to X$, having fibres the spaces $\mathbb{M}_{d(x)^r, d(x)^s}$. Let us denote by $1 \in L(\widehat{\mathcal{H}})$ the identity operator of $\widehat{\mathcal{H}}$; in analogy with the Cuntz algebra, every spectral subspace $\mathcal{O}_\mathcal{H}^k$, $k \in \mathbb{Z}$, is an inductive limit $\lim_{\to r}(\mathcal{H}^r, \mathcal{H}^{r+k})$, with embeddings $i_r : (\mathcal{H}^r, \mathcal{H}^{r+k}) \hookrightarrow (\mathcal{H}^{r+1}, \mathcal{H}^{r+1+k})$, $i_r(t) := t \otimes 1$ (by [6, Proposition 1.8], for every $t \in (\mathcal{H}^r, \mathcal{H}^s)$ there are $f \in C_0(X)$, $t' \in (\mathcal{H}^r, \mathcal{H}^s)$ such that $t = ft'$, thus $t \otimes 1 = t' \otimes f$ is actually a *compact* operator belonging to $(\mathcal{H}^{r+1}, \mathcal{H}^{s+1})$).

The following result has been proved in the abstract setting of certain tensor $C^*$-categories in [26, Theorem 4.1] for 'locally trivial objects', but previously (and independently) by Roberts [24] in the case of vector bundles. As a preliminary remark, note that if $\mathcal{H} \simeq X \times \mathbb{C}^d$, then $\mathcal{O}_\mathcal{H} \simeq C_0(X) \otimes \mathcal{O}_d$.

**Proposition 2.** *Let $X$ be a locally compact Hausdorff space, $\mathcal{H} \to X$ a Hilbert bundle. Then $\mathcal{O}_{\mathcal{H}}$ is a continuous bundle of Cuntz algebras over $X$. If $\mathcal{E} \to X$ is a rank $d$ vector bundle, then $\mathcal{O}_{\mathcal{E}}$ is an $\mathcal{O}_d$-bundle (i.e., $\mathcal{O}_{\mathcal{E}}$ is locally trivial). If $\mathcal{E}$ is a line bundle, then $\mathcal{O}_{\mathcal{E}}$ is a $C(S^1)$-bundle, where $C(S^1)$ is the C*-algebra of continuous functions over the circle.*

**Proof.** Let $t \in (\mathcal{H}^r, \mathcal{H}^s)$; then a vector field $\widehat{t} := \{t_x\} \in \prod_x \mathbb{M}_{d(x)^r, d(x)^s}$ is defined. Since $\mathbb{M}_{d(x)^r, d(x)^s}$ is a vector subspace of $\mathcal{O}_d$, the vector field $\widehat{t}$ can be regarded as an element of $\prod_x \mathcal{O}_{d(x)}$. We consider the set of vector fields $\Theta_{\mathcal{H}} := \left\{\widehat{t} \in \prod_x \mathcal{O}_{d(x)},\ t \in (\mathcal{H}^r, \mathcal{H}^s), r, s \in \mathbb{N}\right\}$; since $\mathcal{O}_{\mathcal{H}}$ is a $C^*$-algebra, it is clear that $\Theta_{\mathcal{H}}$ is a $*$-algebra. Since every $(\mathcal{H}^r, \mathcal{H}^s)$ is a Banach bundle, the norm function $\{x \mapsto \|t_x\|\}$ is continuous and vanishing at infinity; moreover, for every $x \in X$ the set $\left\{t_x, \widehat{t} \in \Theta_{\mathcal{H}}\right\}$ is dense in $\mathcal{O}_{d(x)}$. Thus $\Theta_{\mathcal{H}}$ is a continuous field of $C^*$-algebras, and $\mathcal{O}_{\mathcal{H}}$ is the $C^*$-algebra of continuous, vanishing at infinity sections of $\Theta_{\mathcal{H}}$. Let now $U \subset X$ be a closed set trivializing $\mathcal{H}$; for a generic (non-locally trivial) Hilbert bundle, we can pick $U = \{x\}$, $x \in X$. We consider the corresponding local chart $\pi_U : \mathcal{H}|_U \to U \times \mathbb{C}^{d(x)}$, $x \in U$. Then, a morphism $\pi_U^* : \widehat{\mathcal{H}} \to C(U) \otimes \mathbb{C}^{d(x)}$ is induced, with $\pi_U^*(f\psi) = f|_U\, \pi_U^*(\psi)$, $f \in C_0(X)$, $\psi \in \widehat{\mathcal{H}}$. By universality of the CP-algebra, a local chart $\widehat{\pi}_U : \mathcal{O}_{\mathcal{H}}|_U \to C(U) \otimes \mathcal{O}_{d(x)}$ is induced. We conclude that if $\mathcal{H}$ is a vector bundle, then $\mathcal{O}_{\mathcal{H}}$ is locally trivial. The proof in the case $d(x) \equiv 1$ is analogue, with $C(S^1)$ playing the role of the Cuntz algebra. $\square$

Classification results for Pimsner algebras of line bundles are given in [26, Proposition 4.3].

**Corollary 3.** *The zero-grade algebra $\mathcal{O}_{\mathcal{H}}^0$ is a continuous bundle with fibres $\mathcal{O}_{d(x)}^0$, $x \in X$. For every $k \in \mathbb{Z}$, $\mathcal{O}_{\mathcal{H}}^k$ is an imprimitivity $C_0(X)$-Hilbert $\mathcal{O}_{\mathcal{H}}^0$-bimodule, and defines a Banach bundle over $X$ with fibres $\mathcal{O}_{d(x)}^k$.*

For every $r \in \mathbb{N}$, there is a $C_0(X)$-algebra isomorphism $(\mathcal{H}^r, \mathcal{H}^r) \simeq \otimes_X^r(\mathcal{H}, \mathcal{H})$, so that the zero-grade bundle $\mathcal{O}_{\mathcal{H}}^0$ has a filtration $\mathcal{O}_{\mathcal{H}}^0 \simeq \lim_{\to r} \otimes_X^r(\mathcal{H}, \mathcal{H})$ (see [23, Section 2]). This implies that for every $s \in \mathbb{N}$ there is an isomorphism $\mathcal{O}_{\mathcal{H}}^0 \simeq (\mathcal{H}^s, \mathcal{H}^s) \otimes_X \mathcal{O}_{\mathcal{H}}^0$. Now, for every $r \in \mathbb{N}$ we have that $\widehat{\mathcal{H}^r}$ is a $C_0(X)$-Hilbert $(\mathcal{H}^r, \mathcal{H}^r)$-$C_0(X)$-bimodule. We consider the external tensor product $\widehat{\mathcal{H}^r} \widehat{\otimes}_X \mathcal{O}_{\mathcal{H}}^0$ (in the sense of (4)), which is a $C_0(X)$-Hilbert $((\mathcal{H}^r, \mathcal{H}^r) \otimes_X \mathcal{O}_{\mathcal{E}}^0)$-$\mathcal{O}_{\mathcal{H}}^0$-bimodule; by the above isomorphism for $\mathcal{O}_{\mathcal{H}}^0$, it follows that $\widehat{\mathcal{H}^r} \otimes_X \mathcal{O}_{\mathcal{H}}^0$ has a structure of $C_0(X)$-Hilbert $\mathcal{O}_{\mathcal{H}}^0$-bimodule. Thus (see [23, Proposition 2.3]), there are natural isomorphisms of $C_0(X)$-Hilbert $\mathcal{O}_{\mathcal{H}}^0$-bimodules

$$\delta_r : \widehat{\mathcal{H}^r} \widehat{\otimes}_X \mathcal{O}_{\mathcal{H}}^0 \to \mathcal{O}_{\mathcal{H}}^r, \quad \delta_r(\psi \otimes a) := \psi a, \tag{5}$$

such that $\delta_r((a_0\psi) \otimes (a'a)) = (a_0 \otimes a') \cdot (\psi a)$, $a_0 \in (\mathcal{H}^r, \mathcal{H}^r)$, $a' \in \mathcal{O}_{\mathcal{H}}^0$ (we regard at $a_0 \otimes a'$ as an element of $\mathcal{O}_{\mathcal{H}}^0$). Note that $\mathcal{O}_{\mathcal{H}}^r \simeq \widehat{\mathcal{H}^r} \widehat{\otimes}_X \mathcal{O}_{\mathcal{H}}^0$ is isomorphic to $\widehat{\mathcal{H}^r} \otimes_X \mathcal{O}_{\mathcal{H}}^0$

only as a right Hilbert $\mathcal{O}^0_{\mathcal{H}}$-module. Moreover, there are isomorphisms of $C_0(X)$-Hilbert $\mathcal{O}^0_{\mathcal{H}}$-bimodules

$$\delta_{r,s} : \mathcal{O}^r_{\mathcal{H}} \otimes_{\mathcal{O}^0_{\mathcal{H}}} \mathcal{O}^s_{\mathcal{H}} \to \mathcal{O}^{r+s}_{\mathcal{H}}, \quad \delta_{r,s}(t \otimes t') := tt', \tag{6}$$

where $\otimes_{\mathcal{O}^0_{\mathcal{H}}}$ denotes the internal tensor product with coefficients in $\mathcal{O}^0_{\mathcal{H}}$.

It is well known that the space $\mathbf{end}_1\mathcal{O}_d$ of unital endomorphisms of $\mathcal{O}_d$ is homeomorphic to the unitary group $U\mathcal{O}_d$. Let $M(\mathcal{O}_{\mathcal{E}})$ denote the multiplier algebra of $\mathcal{O}_{\mathcal{E}}$. If $u \in M(\mathcal{O}_{\mathcal{E}})$ is unitary, then by universality of the CP-algebra the map $\left\{\psi \mapsto u\psi,\ \psi \in \widehat{\mathcal{E}}\right\}$ extends to a $C_0(X)$-endomorphism $\rho_u \in \mathbf{end}_X\mathcal{O}_{\mathcal{E}}$. Viceversa, let us consider a set $\left\{\psi_i\right\}$ of generators for $\widehat{\mathcal{E}}$. If $\rho \in \mathbf{end}_X\mathcal{O}_{\mathcal{E}}$, then the net $\left\{\sum_i \rho(\psi_i)\psi_i^*\right\}$ converges in the strict topology to a unitary of $M(\mathcal{O}_{\mathcal{E}})$. Thus, we established a homeomorphism between the space of $C_0(X)$-endomorphisms of $\mathcal{O}_{\mathcal{E}}$ (endowed with the pointwise convergence topology) and the unitary group of $M(\mathcal{O}_{\mathcal{E}})$ (endowed with the norm topology).

**Example 4.** Let $\theta \in L(\widehat{\mathcal{E}}^2)$ denote the flip $\theta(\psi' \otimes \psi) := \psi \otimes \psi'$, $\psi, \psi' \in \widehat{\mathcal{E}}$. It is clear that $\theta = \theta^{-1} = \theta^*$ is a unitary in $M(\mathcal{O}_{\mathcal{E}})$. We define the **canonical endomorphism** $\sigma := \rho_\theta \in \mathbf{end}_X\mathcal{O}_{\mathcal{E}}$. By construction, $\sigma(\psi) = \theta\psi$, $\psi \in \widehat{\mathcal{E}} \subset \mathcal{O}_{\mathcal{E}}$. Moreover, $\sigma(\psi')\psi = \theta\psi'\psi = \psi\psi'$; since $\mathcal{O}_{\mathcal{E}}$ is generated as a $C$*-algebra by elements of $\widehat{\mathcal{E}}$, we obtain $\sigma(t)\psi = \psi t$, $\psi \in \widehat{\mathcal{E}}$, $t \in \mathcal{O}_{\mathcal{E}}$.

## 3. $KK(X; -)$-classes for $\mathcal{O}_{\mathcal{E}}$

Let $X$ be a locally compact, paracompact Hausdorff space, $\mathcal{F}$ an $\mathcal{O}_d$-bundle over $X$. We denote by $\pi_x : \mathcal{F} \to \mathcal{O}_d$, $x \in X$, the epimorphisms associated with $\mathcal{F}$ as a continuous bundle. Suppose now that there is an action $\gamma^X : \mathbb{T} \to \mathbf{aut}_X\mathcal{F}$ such that $\gamma_z \circ \pi_x = \pi_x \circ \gamma^X_z$, $x \in X$, $z \in \mathbb{T}$, where $\gamma : \mathbb{T} \to \mathbf{aut}\mathcal{O}_d$ is the usual circle action over $\mathcal{O}_d$. We say in this case that $\mathcal{F}$ is a *graded $\mathcal{O}_d$-bundle*. Since the circle action on $\mathcal{O}_d$ is semi-saturated, it is easy to verify that $\mathcal{F}$ is semi-saturated (in fact, if $U \subseteq X$ is a trivializing open subset for $\mathcal{F}$ then the restriction $\mathcal{F}_U \simeq C_0(U) \otimes \mathcal{O}_d$ is clearly semi-saturated, and we can extend the generators of $\mathcal{F}_U$ to elements of $\mathcal{F}$, by using cutoff functions); it also clear that $\mathcal{F}^1$ is full, thus there is an isomorphism $(\mathcal{F}, \mathbb{Z}) \simeq (\mathcal{O}_{\mathcal{F}^1}, \mathbb{Z})$. If $\mathcal{E} \to X$ is a rank $d$ vector bundle then $\mathcal{O}_{\mathcal{E}}$ is a graded $\mathcal{O}_d$-bundle, so that there is an isomorphism $(\mathcal{O}_{\mathcal{E}}, \mathbb{Z}) \simeq (\mathcal{O}_{\mathcal{O}^1_{\mathcal{E}}}, \mathbb{Z})$.

If $X$ is $\sigma$-compact and metrisable then $\mathcal{F}$ is separable and $\sigma$-unital, and $\mathcal{F}^1$ is countably generated as a Hilbert $\mathcal{F}^0$-bimodule. Moreover, $\mathcal{F}^0$ acts on the left over $\mathcal{F}^1$ by elements of $K(\mathcal{F}^1) \simeq \mathcal{F}^0$. Thus, we define

$$\delta_1(\mathcal{F}) := [(\mathcal{F}^1, 0)] \in KK(X; \mathcal{F}^0). \tag{7}$$

With an abuse of notation, in the sequel we will denote by $\delta_1(\mathcal{F})$ also the class of $(\mathcal{F}^1, 0)$ in $KK_0(\mathcal{F}^0, \mathcal{F}^0)$ obtained by *forgetting* the $C_0(X)$-structure. Since $\mathcal{F}^1$ is an imprimitivity bimodule, we find that $\delta_1(\mathcal{F})$ is invertible; thus, the Kasparov product by $\delta_1(\mathcal{F})$ defines an automorphism on $KK_0(\mathcal{A}, \mathcal{F}^0)$ for every $C$*-algebra $\mathcal{A}$. In particular, $\delta_1(\mathcal{F}) \in \mathbf{aut}K_0(\mathcal{F}^0)$. From [23, Theorem 4.9] and the isomorphism $\mathcal{F} \simeq \mathcal{O}_{\mathcal{F}^1}$, for every $C$*-algebra $\mathcal{A}$ we obtain the following exact sequence:

$$\begin{tikzcd} KK_0(\mathcal{A}, \mathcal{F}^0) \arrow[r, "1-\delta_1(\mathcal{F})"] & KK_0(\mathcal{A}, \mathcal{F}^0) \arrow[r, "i_0"] & KK_0(\mathcal{A}, \mathcal{F}) \arrow[d, "\delta_0"] \\ KK_1(\mathcal{A}, \mathcal{F}) \arrow[u, "\delta_1"] & KK_1(\mathcal{A}, \mathcal{F}^0) \arrow[l, "i_1"'] & KK_1(\mathcal{A}, \mathcal{F}^0) \arrow[l, "1-\delta_1(\mathcal{F})"'] \end{tikzcd} \tag{8}$$

where $i_*$ are the morphisms induced by the inclusion $\mathcal{F}^0 \hookrightarrow \mathcal{F}$, and $\delta_*$ are the connecting maps induced by the *KK*-equivalence between $\mathcal{F}^0$, $\mathcal{T}_{\mathcal{F}^1}$ (see [23, Theorem 4.4]).

We introduce a notation. Let $\mathcal{K}$ be the $C$*-algebra of compact operators, $\mathcal{F}$, $\mathcal{B}$ graded $\mathcal{O}_d$-bundles such that there is a $C_0(X)$-algebra isomorphism $\alpha : \mathcal{F}^0 \otimes \mathcal{K} \to \mathcal{B}^0 \otimes \mathcal{K}$; then, by functoriality a ring isomorphism $\alpha_* : KK(X; \mathcal{F}^0) \to KK(X; \mathcal{B}^0)$ is defined. We write

$$\delta(\mathcal{F}) = \delta(\mathcal{B}) \quad \Leftrightarrow \quad \alpha_* \delta_1(\mathcal{F}) = \delta_1(\mathcal{B});$$

note that we used the stability of $KK(X; -, -)$, so that we identified $[(\mathcal{F}^1, 0)] \in KK(X; \mathcal{F}^0)$ with $[(\mathcal{F}^1 \widehat{\otimes} \mathcal{K}, 0)] \in KK(X; \mathcal{F}^0 \otimes \mathcal{K})$. The tensor product of $\mathcal{F}^1$ by $\mathcal{K}$ is intended as the external tensor product of Hilbert bimodules. Note that $\alpha_* \delta_1(\mathcal{F}) = [((\mathcal{F}^1 \widehat{\otimes} \mathcal{K})_\alpha, 0)]$, where $(\mathcal{F}^1 \widehat{\otimes} \mathcal{K})_\alpha$ is the pullback bimodule. If there is a graded isomorphism $\beta : (\mathcal{F} \otimes \mathcal{K}, \mathbb{Z}) \to (\mathcal{B} \otimes \mathcal{K}, \mathbb{Z})$, then $\beta(\mathcal{F}^1 \widehat{\otimes} \mathcal{K}) = \mathcal{B}^1 \widehat{\otimes} \mathcal{K}$ and $\delta_1(\mathcal{B}) = \alpha_* \delta_1(\mathcal{F})$, with $\alpha := \beta|_{\mathcal{F}^0 \otimes \mathcal{K}}$; thus, $\delta(\mathcal{B}) = \delta(\mathcal{F})$.

**Theorem 5.** *Let $\mathcal{E} \to X$ be a rank $d$ vector bundle, $i : RK^0(X) \to KK(X; \mathcal{O}_{\mathcal{E}}^0)$ the structure morphism* (2). *Then $\delta_1(\mathcal{O}_{\mathcal{E}}) = i[\mathcal{E}]$.*

**Proof.** Let $I\mathcal{O}_{\mathcal{E}}^0 := C_0[0,1] \otimes \mathcal{O}_{\mathcal{E}}^0$. In order to prove the theorem, it suffice to verify that $\mathcal{O}_{\mathcal{E}}^1$ is homotopic to $\widehat{\mathcal{E}} \otimes_X \mathcal{O}_{\mathcal{E}}^0$, i.e. that there is $(\mathcal{M}, 0) \in KK(X; \mathcal{O}_{\mathcal{E}}^0, I\mathcal{O}_{\mathcal{E}}^0)$ such that $\pi_0(\mathcal{M}, 0) = (\mathcal{O}_{\mathcal{E}}^1, 0)$, $\pi_1(\mathcal{M}, 0) = (\widehat{\mathcal{E}} \otimes_X \mathcal{O}_{\mathcal{E}}^0, 0)$ (where $\pi_s : KK(X; \mathcal{O}_{\mathcal{E}}^0, I\mathcal{O}_{\mathcal{E}}^0) \to KK(X; \mathcal{O}_{\mathcal{E}}^0)$ denotes the evaluation morphism over $s \in [0,1]$). Let $\theta$ be the symmetry associated with the canonical endomorphism $\sigma \in \mathbf{end}_X \mathcal{O}_{\mathcal{E}}$ (Remark 4); if $\psi \in \widehat{\mathcal{E}}$, $a, a' \in \mathcal{O}_{\mathcal{E}}^0$, then

$$\sigma(a)\psi a' = \psi a a'. \tag{9}$$

We denote by $\mathcal{O}^1_{\mathcal{E}}(\sigma)$ the Hilbert bimodule coinciding with $\mathcal{O}^1_{\mathcal{E}}$ as a right Hilbert $\mathcal{O}^0_{\mathcal{E}}$-module, and left action $a, t \mapsto \sigma(a)t$, $t \in \mathcal{O}^1_{\mathcal{E}}$. (9) implies that $\mathcal{O}^1_{\mathcal{E}}(\sigma)$ is isomorphic to $\widehat{\mathcal{E}} \otimes_X \mathcal{O}^0_{\mathcal{E}}$ as a $C_0(X)$-Hilbert $\mathcal{O}^0_{\mathcal{E}}$-bimodule. Now, there is a homotopy $(u_s)_s$, $s \in [0,1]$ in the unitary group of $L(\widehat{\mathcal{E}}^2) \subset M(\mathcal{O}^0_{\mathcal{E}})$, connecting $\theta$ with 1 (see [28, Proposition 4.2.7]). We denote by

$$(\rho_s)_s \ : \ \rho_s(\psi) := u_s \psi, \quad \psi \in \widehat{\mathcal{E}}$$

the corresponding homotopy in the space of endomorphisms of $\mathcal{O}_{\mathcal{E}}$; since every $\rho_s$ is induced by a unitary of $M(\mathcal{O}^0_{\mathcal{E}})$, we find that $\mathcal{O}^0_{\mathcal{E}}$ is $\rho_s$-stable for every $s \in [0,1]$. By construction, $\rho_0 = id$, $\rho_1 = \sigma$. We then consider the space $\mathcal{M} := C([0,1], \mathcal{O}^1_{\mathcal{E}})$ of norm continuous maps from $[0,1]$ into $\mathcal{O}^1_{\mathcal{E}}$, endowed with the Hilbert $\mathcal{O}^0_{\mathcal{E}}$-$I\mathcal{O}^0_{\mathcal{E}}$-bimodule structure

$$(a\psi)\ (s) := \rho_s(a)\psi(s), \quad (\psi b)\ (s) := \psi(s)b(s), \quad \left\langle \psi, \psi' \right\rangle (s) := \psi^*(s)\psi'(s),$$

$\psi, \psi' \in \mathcal{M}$, $a \in \mathcal{O}^0_{\mathcal{E}}$, $b \in I\mathcal{O}^0_{\mathcal{E}}$, $s \in [0,1]$. The left $\mathcal{O}^0_{\mathcal{E}}$-module action over $\mathcal{M}$ realizing the homotopy is by elements of $K(\mathcal{M}) \simeq I\mathcal{O}^0_{\mathcal{E}}$, thus $(\mathcal{M}, 0) \in KK(X; \mathcal{O}^0_{\mathcal{E}}, I\mathcal{O}^0_{\mathcal{E}})$. Moreover, by construction $\pi_0(\mathcal{M}, 0) = (\mathcal{O}^1_{\mathcal{E}}, 0) = \delta_1(\mathcal{O}_{\mathcal{E}})$, $\pi_1(\mathcal{M}, 0) = (\mathcal{O}^1_{\mathcal{E}}(\sigma), 0) = i[\mathcal{E}]$. □

**Remark 6.** Let $X$ be paracompact. The $K$-theory of $\mathcal{O}^0_{\mathcal{E}}$ can be obtained in the following way: as a first step, note that for every $r \in \mathbb{N}$ we find $(\mathcal{E}^r, \mathcal{E}^r) \otimes \mathcal{K} \simeq C_0(X) \otimes \mathcal{K}$ (in fact, $(\mathcal{E}^r, \mathcal{E}^r) \otimes \mathcal{K}$ is a $\mathcal{K}$-bundle with trivial Dixmier–Douady class [11, Section 10]). Thus we have the inductive limit

$$K_0(\mathcal{O}^0_{\mathcal{E}}) = K^0(X) \stackrel{i_0^*}{\to} K^0(X) \stackrel{i_1^*}{\to} K^0(X) \stackrel{i_2^*}{\to} \cdots$$

where $i_r^*$ is the adjoint map of the inclusion $i_r : (\mathcal{E}^r, \mathcal{E}^r) \hookrightarrow (\mathcal{E}^{r+1}, \mathcal{E}^{r+1})$, $i_r(t) := t \otimes 1$. We consider the morphism

$$j : K^0(X) \to K_0(\mathcal{O}^0_{\mathcal{E}}), \quad j := \lim_{\to r} i_r^*. \tag{10}$$

Let $1_X \in K^0(X)$ denote the class of the trivial line bundle over $X$; then $1 = j(1_X) \in K_0(\mathcal{O}^0_{\mathcal{E}})$ is the class of the identity of $\mathcal{O}^0_{\mathcal{E}}$, and we find

$$i[\mathcal{E}] \cdot 1 = j[\mathcal{E}] \in K_0(\mathcal{O}^0_{\mathcal{E}}). \tag{11}$$

**Proposition 7.** *Let $\mathcal{E} \to X$ be a rank $d$ vector bundle. If $\mathcal{E}' \to X$ is a rank $d$ vector bundle with $\delta(\mathcal{O}_{\mathcal{E}}) = \delta(\mathcal{O}_{\mathcal{E}'})$, then $i[\mathcal{E}] = i[\mathcal{E}']$. In particular, $i[\mathcal{E}] = i[\mathcal{E}']$ if there* *is a $C_0(X)$-isomorphism $(\mathcal{O}_{\mathcal{E}} \otimes \mathcal{K}, \mathbb{Z}) \to (\mathcal{O}_{\mathcal{E}'} \otimes \mathcal{K}, \mathbb{Z})$. If the structure morphism $i : RK^0(X) \to KK(X; \mathcal{O}^0_{\mathcal{E}})$ is injective, then $[\mathcal{E}] = [\mathcal{E}'] \in RK^0(X)$.*

**Proof.** We denote by $i' : RK^0(X) \to KK(X; \mathcal{O}^0_{\mathcal{E}'})$ the structure morphism (2). Let $\alpha_* : KK(X; \mathcal{O}^0_{\mathcal{E}'}) \to KK(X; \mathcal{O}^0_{\mathcal{E}})$ be the ring isomorphism induced by an isomorphism between $\mathcal{O}^0_{\mathcal{E}} \otimes \mathcal{K}$, $\mathcal{O}^0_{\mathcal{E}'} \otimes \mathcal{K}$. Then, by (3) we find $i[\mathcal{E}] = \delta_1(\mathcal{O}_{\mathcal{E}}) = \alpha_* \delta_1(\mathcal{O}_{\mathcal{E}'}) = \alpha_* i'[\mathcal{E}'] = i[\mathcal{E}']$. □

From the previous proposition it is clear that the computation of the group $\ker i \subseteq RK^0(X)$ is of interest. A way to check the injectivity of $i$ is to regard at $i[\mathcal{E}]$ as an automorphism of $K_0(\mathcal{O}^0_{\mathcal{E}})$: in fact, if $X$ is compact we find $i[\mathcal{E}] \cdot 1 = j[\mathcal{E}] \in K_0(\mathcal{O}^0_{\mathcal{E}})$ (see (10), (11)). If $j$ is injective, we obtain

$$i[\mathcal{E}] = i[\mathcal{E}'] \;\Rightarrow\; j[\mathcal{E}] = j[\mathcal{E}'] \;\Rightarrow\; [\mathcal{E}] = [\mathcal{E}'] \in K^0(X).$$

Thus, we proved:

**Corollary 8.** *Let $X$ be a compact Hausdorff space, $\mathcal{E} \to X$ a vector bundle such that the morphism $j : K^0(X) \to K_0(\mathcal{O}^0_{\mathcal{E}})$ is injective. If $\mathcal{E}' \to X$ is a vector bundle with a $C_0(X)$-isomorphism $(\mathcal{O}_{\mathcal{E}} \otimes \mathcal{K}, \mathbb{Z}) \to (\mathcal{O}_{\mathcal{E}'} \otimes \mathcal{K}, \mathbb{Z})$, then $[\mathcal{E}] = [\mathcal{E}'] \in K^0(X)$.*

**Remark 9.** If $X = \bullet$ reduces to a single point, then $\mathcal{E} \simeq \mathbb{C}^d$, $RK^0(\bullet) = \mathbb{Z}$, and $KK(\bullet, \mathcal{O}^0_{\mathcal{E}}) = KK_0(\mathcal{O}^0_d, \mathcal{O}^0_d) = \textbf{end}\ \mathbb{Z}\left[\frac{1}{d}\right]$. Thus we find the structure morphism $i : \mathbb{Z} \to \textbf{end}\ \mathbb{Z}\left[\frac{1}{d}\right]$, where $i(k)$ is the multiplication by $k$ in $\mathbb{Z}\left[\frac{1}{d}\right]$ (see [4, 10.11.8; 9]). In particular, $i[\mathcal{E}]$ is the multiplication by $d$ (that is an automorphism on $\mathbb{Z}\left[\frac{1}{d}\right]$). More generally, let $X$ be compact and suppose that $\mathcal{O}^0_{\mathcal{E}} \simeq C(X) \otimes \mathcal{O}^0_d$ (for example, this is always true if $X$ is a $n$-sphere, see [25, Theorem 1.15]). If $K^0(X)$ is finitely generated, we can apply the Kunneth Theorem [4, Section 23] and obtain $K_0(\mathcal{O}^0_{\mathcal{E}}) \simeq K^0(X) \otimes \mathbb{Z}\left[\frac{1}{d}\right]$, $KK_0(\mathcal{O}^0_{\mathcal{E}}, \mathcal{O}^0_{\mathcal{E}}) = \textbf{end}\left(K^0(X) \otimes \mathbb{Z}\left[\frac{1}{d}\right]\right)$: so that $i[\mathcal{E}]$ is the multiplication by the class $j[\mathcal{E}] = [\mathcal{E}] \otimes 1 \in K_0(\mathcal{O}^0_{\mathcal{E}})$.

**Proposition 10.** *Let $X$ be a (compact) finite-dimensional CW-complex, $\mathcal{E}, \mathcal{E}' \to X$ rank $d$ vector bundles with $[\mathcal{E}] = [\mathcal{E}'] \in K^0(X)$. Then, $\delta(\mathcal{O}_{\mathcal{E}}) = \delta(\mathcal{O}_{\mathcal{E}'})$, and there is a $C(X)$-isomorphism $(\mathcal{O}_{\mathcal{E}}, \mathbb{Z}) \to (\mathcal{O}_{\mathcal{E}'}, \mathbb{Z})$.*

**Proof.** Since $X$ is compact, every $\mathcal{O}^k_{\mathcal{E}}$, $k \in \mathbb{Z}$, is finitely generated and projective as a right Hilbert $\mathcal{O}^0_{\mathcal{E}}$-module. Because of the isomorphism $(\mathcal{O}_{\mathcal{E}}, \mathbb{Z}) \simeq (\mathcal{O}_{\mathcal{O}^1_{\mathcal{E}}}, \mathbb{Z})$, it suffice to prove that there is an isomorphism $(\mathcal{O}_{\mathcal{O}^1_{\mathcal{E}'}}, \mathbb{Z}) \simeq (\mathcal{O}_{\mathcal{O}^1_{\mathcal{E}}}, \mathbb{Z})$. In order for that, it suffice to verify that $\mathcal{O}^1_{\mathcal{E}'}$ and some pullback bimodule $(\mathcal{O}^1_{\mathcal{E}})_\alpha$ are outer conjugate as imprimitivity $\mathcal{O}^0_{\mathcal{E}'}$-bimodules (see the remarks in Section 1, and [13, Proposition 3.1]).

For every $r \in \mathbb{N}$, we have $[\mathcal{E}^r] = [\mathcal{E}'^r] \in K^0(X)$ thus, $X$ being a finite-dimensional $CW$-complex, for every $r$ greater than a fixed $r_0 \in \mathbb{N}$ there is an isomorphism $\beta_r : \widehat{\mathcal{E}^r} \to \widehat{\mathcal{E}'^r}$ ([18, VIII. Theorem 1.5]). By functoriality, we have $C^*$-algebra isomorphisms

$$\alpha_r^* : (\mathcal{E}^r, \mathcal{E}^r) \to (\mathcal{E}'^r, \mathcal{E}'^r), \ \ \alpha_r^*(\theta_{\psi,\varphi}) := \theta_{\beta_r(\psi),\beta_r(\varphi)},$$

$$\alpha_r : \mathcal{O}^0_{\mathcal{E}} \to \mathcal{O}^0_{\mathcal{E}'}, \ \ \alpha_r(a_1 \otimes \ldots \otimes a_s \otimes 1 \ldots) := \alpha_r^*(a_1) \otimes \ldots \otimes \alpha_r^*(a_s) \otimes 1 \ldots,$$

where $\psi, \varphi \in \widehat{\mathcal{E}^r}$, $a_1, \ldots, a_s \in (\mathcal{E}^r, \mathcal{E}^r)$. Note that

$$\beta^r(a_0\psi) = \alpha_r^*(a_0)\beta_r(\psi), \quad \alpha_r(a_0 \otimes a') = \alpha_r^*(a_0) \otimes \alpha_r(a'),$$

$a_0 \in (\mathcal{E}^r, \mathcal{E}^r)$, $a' \in \mathcal{O}^0_{\mathcal{E}}$ (we regard at $a_0 \otimes a'$ as an element of $\mathcal{O}^0_{\mathcal{E}} \simeq (\mathcal{E}^r, \mathcal{E}^r) \otimes_X \mathcal{O}^0_{\mathcal{E}}$). In order for more concise notations, we denote $\mathcal{A} := \mathcal{O}^0_{\mathcal{E}'}$, $\gamma_r := \mathcal{O}^r_{\mathcal{E}'}$, so that every $\gamma_r$ is an imprimitivity $\mathcal{A}$-bimodule. By applying (5), we obtain isomorphisms

$$\beta_r^r : \mathcal{O}^r_{\mathcal{E}} \to \gamma_r, \quad \beta_r^r(\psi a) := \beta_r(\psi)\alpha_r(a),$$

$\psi \in \widehat{\mathcal{E}^r}$, $a \in \mathcal{O}^0_{\mathcal{E}}$. Note that the identity $(a_0 \otimes a')\psi a = a_0\psi a' a$ implies

$$\beta_r^r((a_0 \otimes a')\psi a) = \alpha_r^*(a_0)\beta_r(\psi)\alpha_r(a'a) = (\alpha_r^*(a_0) \otimes \alpha_r(a')) \cdot (\beta_r(\psi)\alpha_r(a));$$

thus, we find

$$\beta_r^r(btb') = \alpha_r(b)\beta_r^r(t)\alpha_r(b'), \quad \beta_r^r(t)^*\beta_r^r(t') = \alpha_r(t^*t'),$$

$t, t' \in \mathcal{O}^r_{\mathcal{E}}$, $b, b' \in \mathcal{O}^0_{\mathcal{E}}$. In other terms, the pullback $(\mathcal{O}^r_{\mathcal{E}})_{\alpha_r}$ is isomorphic to $\gamma_r$ as a Hilbert $\mathcal{A}$-bimodule. By applying (6), there are isomorphisms of Hilbert $\mathcal{A}$-bimodules

$$(\mathcal{O}^r_{\mathcal{E}})_{\alpha_{r+1}} \otimes_{\mathcal{A}} (\mathcal{O}^1_{\mathcal{E}})_{\alpha_{r+1}} \simeq (\mathcal{O}^{r+1}_{\mathcal{E}})_{\alpha_{r+1}} \simeq \gamma_{r+1} \simeq \mathcal{O}^1_{\mathcal{E}'} \otimes_{\mathcal{A}} \gamma_r.$$

Since $\gamma_r \simeq (\mathcal{O}^r_{\mathcal{E}})_{\alpha_r} \simeq ((\mathcal{O}^r_{\mathcal{E}})_{\alpha_{r+1}})_\tau$, $\tau := \alpha_r \circ \alpha_{r+1}^{-1} \in \mathbf{aut}_X\mathcal{A}$, by (1) we conclude that $\gamma_r$ is isomorphic to $(\mathcal{O}^r_{\mathcal{E}})_{\alpha_{r+1}}$ as a Hilbert $\mathcal{A}$-bimodule; thus, we obtain that $(\mathcal{O}^1_{\mathcal{E}})_{\alpha_{r+1}}$ and $\mathcal{O}^1_{\mathcal{E}'}$ are outer conjugate. Finally, $\delta_1(\mathcal{O}_{\mathcal{E}'}) = i'[\mathcal{E}'] = i'[\mathcal{E}] = \alpha_{r+1,*}i[\mathcal{E}] = \alpha_{r+1,*}\delta_1(\mathcal{O}_{\mathcal{E}})$, having applied (3); thus, $\delta(\mathcal{O}_{\mathcal{E}}) = \delta(\mathcal{O}_{\mathcal{E}'})$. $\square$

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